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HOMOMORPHISMS OF VECTOR BUNDLES ON CURVES AND PARABOLIC VECTOR BUNDLES ON A SYMMETRIC PRODUCT

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ABSTRACT. Let $S^n(X)$ be the symmetric product of an irreducible smooth complex projective curve X. Given a vector bundle E on X, there is a corresponding parabolic vector bundle $\mathcal{V}_{E*}$ on $S^n(X)$. If E is nontrivial, it is known that $\mathcal{V}_{E*}$ is stable if and only if E is stable. We prove that

$$H^0(S^n(X),\, \mathcal{H}om_{\text{par}}(\mathcal{V}_{E*}\,,\mathcal{V}_{F*})) = H^0(X,\, F\otimes E^\vee)\oplus(H^0(X,\, F)\otimes H^0(X,\, E^\vee)).$$

As a consequence, the map from a moduli space of vector bundles on X to the corresponding moduli space of parabolic vector bundles on $S^n(X)$ is injective.

1. Introduction

Let X be an irreducible smooth complex projective curve. Fixing an integer $n \geq 2$, let $S^n(X)$ be the n–fold symmetric product of X. Let $D \subset S^n(X)$ be the reduced irreducible divisor parametrizing nonreduced effective divisors of X of length n. Let

$$q_1 \,:\, S^n(X)\times X \longrightarrow S^n(X) \quad \text{and} \quad q_2 \,:\, S^n(X)\times X \longrightarrow X$$

be the natural projections. The tautological hypersurface on $S^n(X)\times X$ will be denoted by Δ. Given a vector bundle E on X, define the vector bundle

$$\mathcal{F}(E) \,:=\, q_{1*}(\mathcal{O}_\Delta \otimes_{\mathcal{O}_{S^n(X)\times X}} q_2^*E) \longrightarrow S^n(X).$$

This vector bundle $\mathcal{F}(E)$ has a natural parabolic structure over the divisor D; the parabolic weights are 0 and 1/2. (See [BL] for the construction of the parabolic structure.) This parabolic vector bundle will be denoted by $\mathcal{V}_{E*}$.

The parabolic vector bundle $\mathcal{V}_{E*}$ is semistable if and only if the vector bundle E is semistable [BL, Lemma 1.2]. If E is not the trivial vector bundle, then $\mathcal{V}_{E*}$ is stable if and only if E is stable [BL, Theorem 1.3].

Therefore, a morphism from a moduli space of vector bundles on X to a moduli space of parabolic vector bundles on $S^n(X)$ is obtained by sending any E to $\mathcal{V}_{E*}$.

Our aim here is to show that the above morphism is injective.

For two parabolic vector bundles V_* and W_* on $S^n(X)$ with D as the parabolic divisor and underlying vector bundles V and W respectively, let $\mathcal{H}om_{\text{par}}(V_*\,,V_*)$ be the vector bundle on $S^n(X)$ defined by the sheaf of homomorphisms from V to W preserving the parabolic structures.

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We prove the following (see Corollary 3.4):

Theorem 1.1. *Let E and F be stable vector bundles over X with*

$$\text{degree}(E)/\text{rank}(E) \,=\, \text{degree}(F)/\text{rank}(F).$$

Then

$$H^0(S^n(X),\, \mathcal{H}om_{\text{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*})) \,=\, 0$$

if $E \neq F$, and

$$\dim H^0(S^n(X),\, \mathcal{H}om_{\text{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*})) \,=\, 1$$

if $E = F$.

In fact we show that for any vector bundles E and F on X,

$$H^0(S^n(X),\, \mathcal{H}om_{\text{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*})) \,=\, H^0(X,\, F\otimes E^\vee)\oplus (H^0(X,\, F)\otimes H^0(X,\, E^\vee)).$$

(See Theorem 3.3.)

2. Invariants of the tensor product

Let X be an irreducible smooth projective curve defined over $\mathbb{C}$. Take any integer $n \geq 2$. For any $i \in [1\,, n]$, let

$$p_i \,:\, X^n \,\longrightarrow\, X \tag{2.1}$$

be the projection to the i–th factor. The group of permutations of $\{1\,, \cdots, n\}$ will be denoted by $\Sigma(n)$. There is a natural action of it on X^n,

$$X^n \times \Sigma(n) \,\longrightarrow\, X^n, \tag{2.2}$$

that permutes the factors. If V_0 is a vector bundle on X, the above action of $\Sigma(n)$ on X^n has a natural lift to an action of $\Sigma(n)$ on the vector bundle

$$\mathcal{V}_0 \,:=\, \bigoplus_{i=1}^{n} p_i^* V_0 \,\longrightarrow\, X^n$$

which simply permutes the factors in the direct sum.

Take two algebraic vector bundles V and W over X. Define

$$\mathcal{V} \,:=\, \bigoplus_{i=1}^{n} p_i^* V \quad \text{and} \quad \mathcal{W} \,:=\, \bigoplus_{i=1}^{n} p_i^* W.$$

As noted above, $\mathcal{V}$ and $\mathcal{W}$ are equipped with an action of $\Sigma(n)$. The Künneth formula says that

$$H^0(X^n,\, \mathcal{V}\otimes\mathcal{W}) \,=\, \bigoplus_{i,j=1}^{n} H^0(X^n,\, p_i^*V\otimes p_j^*W).$$

Using the projection formula, we have

$$H^0(X^n,\, p_i^*V\otimes p_i^*W) \,=\, H^0(X,\, V\otimes W)\,, \tag{2.3}$$

and if $i \neq j$, then

$$H^0(X^n,\, p_i^*V\otimes p_j^*W) \,=\, H^0(X,\, V)\otimes H^0(X,\, W). \tag{2.4}$$

Using these we get an embedding

(2.5)

$$\Phi \,:\, H^0(X,\, V\otimes W)\oplus (H^0(X,\, V)\otimes H^0(X,\, W)) \,\longrightarrow\, \bigoplus_{i,j=1}^{n} H^0(X^n,\, p_i^*V\otimes p_j^*W)$$

$$= H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})$$

that sends any $s\,\in\, H^0(X,\, V\otimes W)$ to

$$\bigoplus_{i=1}^{n} p_i^* s \,\in\, \bigoplus_{i=1}^{n} H^0(X^n,\, p_i^*(V\otimes W)) \,\subset\, \bigoplus_{i,j=1}^{n} H^0(X^n,\, p_i^*V\otimes p_j^*W)$$

and sends any $u\otimes t\,\in\, H^0(X,\, V)\otimes H^0(X,\, W)$ to

$$\sum_{(i,j)\,\in\,[1,n]\times[1,n];i\neq j} (p_i^*u)\otimes (p_j^*t) \,\in\, \bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n,\, p_i^*V\otimes p_j^*W)$$

$$\subset\, \bigoplus_{i,j=1}^{n} H^0(X^n,\, p_i^*V\otimes p_j^*W).$$

The actions of $\Sigma(n)$ of $\mathcal{V}$ and $\mathcal{W}$ together produce a linear action of $\Sigma(n)$ on $H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})$. Let

$$H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)} \,\subset\, H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})$$

be the space of invariants.

Proposition 2.1. *The homomorphism* Φ *in* (2.5) *is an isomorphism of*

$$H^0(X,\, V\otimes W)\oplus (H^0(X,\, V)\otimes H^0(X,\, W))$$

with $H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)}$.

Proof. From the construction of Φ it follows immediately that

$$\Phi(H^0(X,\, V\otimes W)\oplus (H^0(X,\, V)\otimes H^0(X,\, W))) \,\subset\, H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)}.$$

Also, Φ is clearly injective.

Consider the decomposition of the vector bundle

$$\mathcal{V}\otimes\mathcal{W} \,=\, (\bigoplus_{i=1}^{n} p_i^*(V\otimes W))\oplus (\bigoplus_{i,j=1;i\neq j}^{n} (p_i^*V)\otimes (p_j^*W)) \tag{2.6}$$

into a direct sum of two vector bundles. Clearly, the action of $\Sigma(n)$ on $\mathcal{V}\otimes\mathcal{W}$ leaves the two direct summands

$$\bigoplus_{i=1}^{n} p_i^*(V\otimes W) \quad \text{and} \quad \bigoplus_{i,j=1;i\neq j}^{n} (p_i^*V)\otimes (p_j^*W) \tag{2.7}$$

in (2.6) invariant.

Since the second subbundle in (2.7) is $\Sigma(n)$–invariant, the subspace

$$\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n,\, (p_i^*V)\otimes (p_j^*W)) \,\subset\, H^0(X^n,\, \mathcal{V}\otimes\mathcal{W}) \tag{2.8}$$

is left invariant by the action of $\Sigma(n)$ on $H^0(X^n,\, \mathcal{V}\otimes\mathcal{W})$.

Let $\mathcal{A}$ be the complex vector space of dimension $n^2 - n$ given by the space of all functions

$$\alpha : \{1, \cdots, n\} \times \{1, \cdots, n\} \longrightarrow \mathbb{C}$$

such that $\alpha(i,i) = 0$ for all $i \in [1, n]$. The permutation action of $\Sigma(n)$ on $\{1, \cdots, n\}$ produces an action of $\Sigma(n)$ on $\mathcal{A}$. Consider the $\Sigma(n)$–invariant subspace

$$\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^*V)\otimes(p_j^*W))$$

in (2.8). From (2.4) it follows that

$$(2.9)\quad (\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^*V)\otimes(p_j^*W)))^{\Sigma(n)} = \mathcal{A}^{\Sigma(n)} \otimes H^0(X, V) \otimes H^0(X, W),$$

where $(\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^*V)\otimes(p_j^*W)))^{\Sigma(n)}$ and $\mathcal{A}^{\Sigma(n)}$ are the spaces of invariants.

The space of invariants $\mathcal{A}^{\Sigma(n)}$ is generated by the function

$$\rho : [1,n] \times [1,n] \longrightarrow \mathbb{C}$$

defined by $(i, j) \longmapsto 1-\delta^i_j$, where $\delta^i_j = 0$ if $i \neq j$ and $\delta^i_i = 1$ for all i. This follows from Burnside's theorem (see [La, p. 648] for Burnside's theorem). Therefore, we have

$$(2.10)\quad \mathcal{A}^{\Sigma(n)} = \mathbb{C}\cdot\rho = \mathbb{C}.$$

From (2.9) and (2.10) we conclude that

$$(2.11)\quad (\bigoplus_{i,j=1;i\neq j}^{n} H^0(X^n, (p_i^*V)\otimes(p_j^*W)))^{\Sigma(n)} = H^0(X, V)\otimes H^0(X, W).$$

In view of (2.6) and (2.11),

(2.12)

$$H^0(X^n, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)} = (\bigoplus_{i=1}^{n} H^0(X^n, p_i^*(V\otimes W)))^{\Sigma(n)} \oplus (H^0(X, V)\otimes H^0(X, W)).$$

Let $\mathcal{B}$ be the complex vector space of dimension n given by the space of all functions

$$\{1, \cdots, n\} \longrightarrow \mathbb{C}.$$

Let $\mathcal{B}_0 = \mathbb{C} \subset \mathcal{B}$ be the line defined by the constant functions. The group $\Sigma(n)$ has a natural action on $\mathcal{B}$. From (2.12) and (2.3),

$$(2.13)\quad H^0(X^n, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)} = (\mathcal{B}^{\Sigma(n)}\otimes H^0(X, V\otimes W))\oplus(H^0(X, V)\otimes H^0(X, W)).$$

It can be shown that

$$\mathcal{B}^{\Sigma(n)} = \mathcal{B}_0,$$

where $\mathcal{B}_0$ is defined above. Indeed, this is an immediate corollary of Burnside's theorem mentioned above. Therefore, from (2.13),

$$H^0(X^n, \mathcal{V}\otimes\mathcal{W})^{\Sigma(n)} = H^0(X, V\otimes W)\oplus(H^0(X, V)\otimes H^0(X, W)).$$

This completes the proof of the proposition. $\square$

3. Homomorphisms of vector bundles and parabolic vector bundles

Let

$$f : X^n \longrightarrow X^n/\Sigma(n) =: S^n(X) \tag{3.1}$$

be the quotient map to the symmetric product of X. Let $E \longrightarrow X$ be a vector bundle. The action of $\Sigma(n)$ on the vector bundle

$$\mathcal{E} := \bigoplus_{i=1}^{n} p_i^* E \tag{3.2}$$

produces an action of $\Sigma(n)$ on the direct image $f_*\mathcal{E}$; the morphism $f_*\mathcal{E} \longrightarrow S^n(X)$ is $\Sigma(n)$–equivariant with $\Sigma(n)$ acting trivially on $S^n(X)$. The invariant direct image

$$\mathcal{V}_E := (f_*\mathcal{E})^{\Sigma(n)} \subset f_*\mathcal{E} \tag{3.3}$$

is a locally free $\mathcal{O}_{S^n(X)}$–module. Using the action of $\Sigma(n)$ on $\mathcal{E}$, a parabolic structure on the vector bundle $\mathcal{V}_E$ is constructed (see [BL, Section 3]). This parabolic vector bundle will be denoted by $\mathcal{V}_{E*}$. We will now quickly recall the description of $\mathcal{V}_{E*}$.

Let

$$D \subset S^n(X)$$

be the reduced irreducible divisor parametrizing all $(z_1, \cdots, z_n)$ such that not all z_i are distinct. The parabolic divisor for $\mathcal{V}_{E*}$ is D. Let

$$\widetilde{D} \subset X^n$$

be the reduced divisor parametrizing all $(z_1, \cdots, z_n)$ such that $z_i \neq z_j$ for some $i, j \in [1, n]$. So, $f(\widetilde{D}) = D$. The action of $\Sigma(n)$ on $\mathcal{E}$ preserves the coherent subsheaf $\mathcal{E} \otimes \mathcal{O}_{X^n}(-\widetilde{D})$. Define the invariant direct image

$$\mathcal{V}'_E := (f_*(\mathcal{E} \otimes \mathcal{O}_{X^n}(-\widetilde{D})))^{\Sigma(n)} \subset f_*(\mathcal{E} \otimes \mathcal{O}_{X^n}(-\widetilde{D})). \tag{3.4}$$

Clearly,

$$\mathcal{V}'_E \subset \mathcal{V}_E. \tag{3.5}$$

The parabolic bundle $\mathcal{V}_{E*}$ is defined as follows: $(\mathcal{V}_E)_{1/2} = \mathcal{V}'_E$ and $(\mathcal{V}_E)_0 = \mathcal{V}_E$ (see [MY]). Therefore, the quasi–parabolic filtration is a 1–step filtration, and it is constructed from (3.5); the parabolic weights are $1/2$ and 0.

Let F be a vector bundle over X. Define the vector bundles

$$\mathcal{F} := \bigoplus_{i=1}^{n} p_i^* F \quad \text{and} \quad \mathcal{V}_F := (f_*\mathcal{F})^{\Sigma(n)}. \tag{3.6}$$

Let $\mathcal{V}_{F*}$ be the parabolic vector bundle on $S^n(X)$, with $\mathcal{V}_F$ as the underlying vector bundle and parabolic structure over D, obtained by substituting F for E in the above construction of $\mathcal{V}_{E*}$. Define $\mathcal{V}'_F \subset \mathcal{V}_E$ as done in (3.4). Let

$$\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}) \subset \mathcal{H}om(\mathcal{V}_{E*}, \mathcal{V}_{F*}) \tag{3.7}$$

be the sheaf of homomorphisms compatible with the parabolic structures [MY], [MS]. We recall that a section T of $\mathcal{H}om(\mathcal{V}_{E*}, \mathcal{V}_{F*}) = \mathcal{V}_{F*} \otimes (\mathcal{V}_{E*})^\vee$ defined over an open subset $U \subset S^n(X)$ lies in $\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})$ if and only if

$$T(\mathcal{V}'_E|_U) \subset \mathcal{V}'_F.$$

Define the vector bundle

$$W_{E,F} := \bigoplus_{i,j=1;i\neq j}^{n} p_i^* F \otimes p_j^* E^\vee \longrightarrow X^n. \tag{3.8}$$

The actions of $\Sigma(n)$ on $\mathcal{E}^\vee$ and $\mathcal{F}$ together define an action of $\Sigma(n)$ on $W_{E,F}$. Define the vector bundle

$$\mathcal{W}_{E,F} := (f_* W_{E,F})^{\Sigma(n)} \longrightarrow S^n(X). \tag{3.9}$$

Lemma 3.1. *Let $\mathcal{V}_{F\otimes E^\vee}$ be the vector bundle on $S^n(X)$ obtained by substituting $F \otimes E^\vee$ for E in the construction of $\mathcal{V}_E$. There is a canonical injective homomorphism of $\mathcal{O}_{S^n(X)}$–modules*

$$H : \mathcal{V}_{F\otimes E^\vee} \oplus \mathcal{W}_{E,F} \longrightarrow \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}),$$

where $\mathcal{W}_{E,F}$ and $\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})$ are defined in (3.9) and (3.7) respectively.

Proof. Consider $\mathcal{E}$ and $\mathcal{F}$, defined in (3.2) and (3.6) respectively, equipped with action of $\Sigma(n)$. From the constructions of $\mathcal{V}_{F\otimes E^\vee}$ and $\mathcal{W}_{E,F}$ it follows that

$$\mathcal{V}_{F\otimes E^\vee} \oplus \mathcal{W}_{E,F} = (f_*(\mathcal{F} \otimes \mathcal{E}^\vee))^{\Sigma(n)}.$$

Note that $\mathcal{F} \otimes \mathcal{E}^\vee = (\bigoplus_{i=1}^n p_i^*(F \otimes E^\vee)) \oplus W_{E,F}$, where $W_{E,F}$ is constructed in (3.8).

Take any nonempty Zariski open subset $U \subset S^n(X)$. Let

$$\phi : \mathcal{E}|_{f^{-1}(U)} \longrightarrow \mathcal{F}|_{f^{-1}(U)} \tag{3.10}$$

be a homomorphism which intertwines the actions of $\Sigma(n)$ on $\mathcal{E}|_{f^{-1}(U)}$ and $\mathcal{F}|_{f^{-1}(U)}$, where f is the quotient map in (3.1). Let

$$\widetilde{D}_U := \widetilde{D} \cap f^{-1}(U)$$

be the divisor on $f^{-1}(U)$. Let

$$\widetilde{\phi} := \phi \otimes \mathrm{Id} : \mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U) \longrightarrow \mathcal{F}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U) \tag{3.11}$$

be the homomorphism, where Id is the identity automorphism of $\mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U)$. The restriction of ϕ to the subsheaf

$$\mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U) \subset \mathcal{E}|_{f^{-1}(U)}$$

clearly coincides with $\widetilde{\phi}$.

Since the action of $\Sigma(n)$ on X^n leaves $\widetilde{D}_U$ invariant, we get an action of $\Sigma(n)$ on $\mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U)$. The actions of $\Sigma(n)$ on $\mathcal{E}|_{f^{-1}(U)}$ and $\mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U)$ together produce an action of $\Sigma(n)$ on $\mathcal{E}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U)$. Similarly, $\mathcal{F}|_{f^{-1}(U)} \otimes \mathcal{O}_{f^{-1}(U)}(-\widetilde{D}_U)$ is equipped with an action of $\Sigma(n)$. Since ϕ in (3.10) is $\Sigma(n)$–equivariant, it follows immediately that the homomorphism $\widetilde{\phi}$ in (3.11) is also $\Sigma(n)$–equivariant. Consequently, ϕ produces a section of $\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})$ over U.

Therefore, we have a homomorphism of $\mathcal{O}_{S^n(X)}$–modules

$$H : \mathcal{V}_{F\otimes E^\vee} \oplus \mathcal{W}_{E,F} \longrightarrow \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}) \tag{3.12}$$

that sends any section ϕ of

$$(f_*(\mathcal{F} \otimes \mathcal{E}^\vee))^{\Sigma(n)} = \mathcal{V}_{F\otimes E^\vee} \oplus \mathcal{W}_{E,F}$$

over some open subset U to the section of

$$\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})|_U$$

constructed above from ϕ. $\square$

Proposition 3.2. *Take two vector bundles E and F on X. The homomorphism*

$$\widehat{H} : H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}) \longrightarrow H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}))$$

given by H in Lemma 3.1 *is an isomorphism.*

Proof. Since H is injective, the corresponding homomorphism

$$\widehat{H} : H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}) \longrightarrow H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}))$$

is also injective. So to prove that $\widehat{H}$ is an isomorphism, it suffices to show that

$$\begin{aligned} \dim H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) \leq{} & \dim H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \\ & + \dim H^0(S^n(X), \mathcal{W}_{E,F}). \end{aligned} \tag{3.13}$$

From the construction of the vector bundle $\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})$ in (3.7) it follows that

$$\mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*}) \subset (f_*(\mathcal{F}\otimes\mathcal{E}^\vee))^{\Sigma(n)} \subset f_*(\mathcal{F}\otimes\mathcal{E}^\vee),$$

where $\mathcal{E}$ and $\mathcal{F}$ are constructed in (3.2) and (3.6) respectively. Consequently,

$$H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) \subset H^0(X^n, f_*(\mathcal{F}\otimes\mathcal{E}^\vee))^{\Sigma(n)}.$$

Hence setting V and W in Proposition 2.1 to be F and $E^\vee$ respectively we conclude that

$$\begin{aligned} H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) &\subset H^0(X^n, f_*(\mathcal{F}\otimes\mathcal{E}^\vee))^{\Sigma(n)} \\ &= H^0(X, F\otimes E^\vee) \oplus (H^0(X, F)\otimes H^0(X, E^\vee)). \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} & H^0(X, F\otimes E^\vee) \oplus (H^0(X, F)\otimes H^0(X, E^\vee)) \\ & \quad \subset H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}). \end{aligned} \tag{3.15}$$

Indeed, $H^0(X, F\otimes E^\vee) \subset H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee})$ and

$$H^0(X, F)\otimes H^0(X, E^\vee) \subset H^0(S^n(X), \mathcal{W}_{E,F}).$$

Combining (3.14) and (3.15),

$$H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) \subset H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \oplus H^0(S^n(X), \mathcal{W}_{E,F}).$$

Therefore, we conclude that the inequality in (3.13) holds. This completes the proof of the proposition. $\square$

Theorem 3.3. *There is a canonical isomorphism*

$$H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) \xrightarrow{\sim} H^0(X, F\otimes E^\vee) \oplus (H^0(X, F)\otimes H^0(X, E^\vee)).$$

Proof. From Proposition 3.2,

$$\begin{aligned} \dim H^0(S^n(X), \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}, \mathcal{V}_{F*})) ={} & \dim H^0(S^n(X), \mathcal{V}_{F\otimes E^\vee}) \\ & + \dim H^0(S^n(X), \mathcal{W}_{E,F}), \end{aligned}$$

and from (3.15),

$$\dim H^0(X,\, F\otimes E^\vee) + \dim(H^0(X,\, F)\otimes H^0(X,\, E^\vee)) \\ \le\ \dim H^0(S^n(X),\, \mathcal{V}_{F\otimes E^\vee}) + \dim H^0(S^n(X),\, \mathcal{W}_{E,F}).$$

Consequently,

$$\dim H^0(X,\, F\otimes E^\vee) + \dim(H^0(X,\, F)\otimes H^0(X,\, E^\vee)) \\ \le\ \dim H^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*})).$$

On the other hand,

$$\dim H^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*}))\ \le \dim H^0(X,\, F\otimes E^\vee) \\ + \dim(H^0(X,\, F)\otimes H^0(X,\, E^\vee))$$

(see (3.14)). Combining these we conclude that

$$h^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*}))\ =\ h^0(X,\, F\otimes E^\vee) + h^0(X,\, F)\cdot h^0(X,\, E^\vee)).$$

Therefore, the subspace

$$H^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*}))$$

in (3.14) coincides with the ambient space

$$H^0(X,\, F\otimes E^\vee)\oplus(H^0(X,\, F)\otimes H^0(X,\, E^\vee)).$$ □

Corollary 3.4. *Let E and F be stable vector bundles over X with*

$$\mathrm{degree}(E)/\mathrm{rank}(E)\ =\ \mathrm{degree}(F)/\mathrm{rank}(F).$$

Then

$$H^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*}))\ =\ 0$$

if $E\,\neq\,F$, and

$$\dim H^0(S^n(X),\, \mathcal{H}om_{\mathrm{par}}(\mathcal{V}_{E*}\,, \mathcal{V}_{F*}))\ =\ 1$$

if $E\,=\,F$.

Proof. If $\mathrm{degree}(F)\,\le\,0$, then $H^0(X,\,F)\,=\,0$ because F is stable. If $\mathrm{degree}(F)\,>\,0$, then $H^0(X,\,E^\vee)\,=\,0$ because $E^\vee$ is stable with $\mathrm{degree}(E^\vee)\,<\,0$.

Therefore, $H^0(X,\,F)\otimes H^0(X,\,E^\vee)\,=\,0$. Hence the corollary follows from Theorem 3.3. □

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